SOLUTIONS OF SOME NON-LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF THE SECOND KIND USING MODIFIED VARIATIONAL ITERATION METHOD

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Abstract: In this paper, Modified Variational Iteration Method is used to solve some nonlinear Volterra Integro-Differential equations of the second kind. With illustrative examples, the method provides a sequence of functions which converges to the exact solution of the problem by using three iterative steps without discretization of the variables. Comparison of the approximate solutions of this method with other methods shows that Modified Variational Iteration scheme is very effective, accurate, more promising and readily implemented.

Keywords: Non-linear Volterra Integro-differential equations, modified variational iteration method

Introduction

Different numerical methods have been used by physical and engineering researchers in solving Volterra Integro-differential equations. Many of these methods have led to accurate and reliable solutions. Brunner (1986) employed high order numerical methods for solving Volterra Integro-differential equations. Day (1967) used trapezoidal rule to devise a numerical method for solving nonlinear volterra integro-differential equation. El-Sayed and Abdul-Aziz (2003) compared Adomian decomposition method and Wavelet-Galerkin method in solving Integro-differential equations. Ghasemi *et al.* (2011) applied He's homotopy perturbation to solve nonlinear Integro-differential equation. Linz (1969) derived a fourth order numerical method for solving nonlinear volterra integro- differential equations of the second kind. Mahmoud and Mahdi (2013) used a numerical method with new orthogonal basis function set to solve nonlinear Volterra-Fredholms integral equations. Maleknejad *et al.* (2011) used hybrid Legendre polynomials with Block-pulse functions to obtain a numerical solution for nonlinear Volterra-Fredholms integro-differential equations. Mehdiyeva *et al.* (2013) suggested some ways in constructing hybrid method for solving nonlinear volterra integral equations of the second kind. Nadjafi and Ghorbani (2009) confirmed that He's homotopy perturbation method is an effective tool for solving nonlinear integral and integro-differential equations. Prakash and Santanu (2015) converted Lane-Emden equations of the first and second kinds into Volterra integro-differential equations and then solve them using Legendre multi-wavelet method. Saadati *et al.* (2008) made a comparison between the variational iteration method and trapezoidal rule in solving linear integro-differential equations. Saeedi, Tari and Masuleh described the operational Tau method for solving nonlinear Volterra Integro-Differential equations of the second kind. Venkatesh *et al.* (2012) applied Legendre wavelet direct method for solving integro-differential equations. Wazwaz, Rach and Duan used Adomian decomposition method for solving the Volterra integral form of the Lane-Emden equations with initial values and boundary conditions. Zhao and Corless (2012) adopted compact finite difference method in solving integro-differential equations.

In this paper, Modified variational iteration method is used to solve some nonlinear Volterra integro-differential equations of the second kind. Some numerical examples are given with their exact and approximate solutions. Tables showing absolute errors and comparison with existing methods are used to demonstrate the accuracy and efficacy of this method.

The Non-Linear Volterra Integro-Differential Equations

Consider the general form of the non-linear, Volterra, Integro-Differential equations of the second kind

$$u^{(n)}(x) = f_i(x) + \sum_{i=0}^{m} \lambda_i \int_o^x (K_i(x,t)F(u(t))dt, u^k(0) = c_k,\ 0 \le k \le (n-1). \qquad (1)$$

$u^{(n)}(x)$ indicate the nth derivatives of $u(x)$, c_k are constants that represent the initial conditions and $F(u(t))$ is non-linear; $u(x)$, $f_i(x)$ are assumed to be real and λ_i are real finite constants, F, f_i, and K_i are continuous functions and u is the unknown function to be determined

Analysis of the Method

Consider the differential equation $Lu + Nu = g(x)$ (2)

where L , N are linear and non linear opertors, $g(x)$ is the non homogeneous term. The correction functional for equation (2) is given as

$$\mathrm{u}_{i+1}(\mathrm{x}) = \mathrm{u}_i(\mathrm{x}) + \int_{\mathrm{a}}^{\mathrm{x}} \lambda(\xi)(Lu_i(\xi) + N\bar{u}_i(\xi) - \mathrm{g}(\xi))\mathrm{d}\xi \qquad (3)$$

λ is a general Lagrange's multiplier, which can be identified optimally via integration by parts and by using a restricted variation.

Setting $Lu_i(\xi) = u_i'(\xi)$. (4)

$$\int_a^x \lambda(\xi)\left(u_i'(\xi)\right)\mathrm{d}\xi = \lambda(\xi)\,u_i(\xi) - \int_a^x \lambda'(\xi)(u_i(\xi))\mathrm{d}\xi. \qquad (5)$$

$$\left.\begin{aligned} \int_a^x \lambda(\xi)\left(u_i''(\xi)\right)\mathrm{d}\xi &= \lambda(\xi)\,u_i'(\xi) - \lambda'(\xi)(u_i(\xi) + \int_a^x \lambda''(\xi)u_i(\xi))\mathrm{d}\xi \\ \int_a^x \lambda(\xi)\left(u_i'''(\xi)\right)\mathrm{d}\xi &= \lambda(\xi)\,u_i''(\xi) - \lambda'(\xi)(u_i'(\xi) + \lambda''(\xi)u_i(\xi) - \int_a^x \lambda'''(\xi)u_i(\xi))\mathrm{d}\xi \\ \int_a^x \lambda(\xi)\left(u_i^{'v}(\xi)\right)\mathrm{d}\xi &= \lambda(\xi)\,u_i'''(\xi) - \lambda'(\xi)(u_i''(\xi) + \lambda''(\xi)u_i'(\xi) - \lambda'''(\xi)u_i(\xi) + \int_a^x \lambda^{'v}(\xi)u_i(\xi))\mathrm{d}\xi \\ &\ \ \vdots \end{aligned}\right\} \qquad (6)$$

The generalized integration by parts is

$$\int_a^x \lambda(\xi)\left(u_i{}^n(\xi)\right)d\xi = \lambda(\xi)\, u_i{}^{n-1}(\xi) - \lambda'(\xi)u_i{}^{n-2}(\xi) + \lambda''(\xi)u_i{}^{n-3}(\xi) - \cdots - (-1)^n \int_a^x \lambda^n(\xi)u_i(\xi))d\xi$$

Where $\lambda(\xi)$ may be a constant or a function, and δ is a restricted value that can behave as a constant, $\bar{u}_i$ is considered as restricted variationand $\delta\bar{u}_i = 0$, where δ is the variational derivative. The extremum condition of u_{i+1} requires that $\delta u_{i+1} = 0$ and this yields the stationary conditions:

$$1 + \lambda|_{\xi=x} = 0,\ \lambda'\big|_{\xi=x} = 0\ \text{, hence } \lambda = -1 \tag{7}$$

The successive approximations $u_{i+1},\ i \geq 0$, of the solution $u(x)$ will be obtained by using selective functions $u_i(x)$. The Non-linear term is expressed in a unique way that gives a better approximation than other numerical methods.

A special case of (1) is

$$u^{(n)}(x) + \lambda \int_a^{h(x)} k(x,t)u^{(p)}(t)\, u^{(m)}(t)dt = g(x): \tag{8}$$

Subject to the initial conditions $u^{(r)}(0) = c_r$ where $c_r,\ r = 0, 1, \dots (n-1)$ are real constant and p, m *are integers* with $p \leq m < n$.

In solving (1), consider the following general functional equation of the form

$$\mathcal{L}u = f + N(u_i)$$

where f is a known analytical function, $N(u_i)$ is the nonlinear operator which is decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \tag{9}$$

$u_j(x)$ are polynomials of x, $N(u_0 + u_1 + u_2 + \cdots) = N(u_0) + \sum_{i=1}^{\infty}\{N(u_0 + u_1 + \cdots + u_i) - N(u_0 + u_1 + \cdots + u_{i-1})\}$

The recurrence relations are defined as

$$\left.\begin{aligned} u_0 &= f \\ u_1 &= N(u_0) \\ u_2 &= N(u_0 + u_1) - N(u_0) \\ u_3 &= N(u_0 + u_1 + u_2) - N(u_0 + u_1) \\ &\ \ \vdots \\ u_{i+1} &= N(u_0 + u_1 + \cdots + u_i) - N(u_0 + u_1 + \cdots + u_{i-1})\quad i = 1, 2, \dots \end{aligned}\right\} \tag{10}$$

Assume a series solution of the form $u = \sum_{i=0}^{\infty} u_i$ (11)

The non-linear term in (3) can be written as $N\tilde{u}_i(\xi) = Nu_i(\xi)$

The nth term approximate solution in (10) is $u_0 + u_1 + \cdots u_{i+1} = N(u_0 + u_1 + \cdots + u_i)$

$u = \sum_{i=0}^{i-1} u_i(x)$

Apply $\mathcal{L}^{-1}$ to the recurrence relation for the determination of the components, the $(n+1)th$ approximation of the exact solution for the unknown function $u(x)$ is obtained as

$$u_{i+1}(x) = N(u_0 + u_1 + \cdots + u_i) - N(u_0 + u_1 + \cdots + u_{i-1}) = \mathcal{L}^{-1}(N(u_0 + u_1 + \cdots + u_i) - \mathcal{L}^{-1}(N(u_0 + u_1 + \cdots + u_{i-1})$$

The solution is constructed as

$$u(x) = \mathcal{L}^{-1}\sum_{i=0}^{i-1} u_n(x),\ n \geq 0 \tag{12}$$

$$u_{i+1}(x) = u_i(x) + \int_a^x \lambda(\xi)(Lu_i(\xi) + N\tilde{u}_i(\xi) - g(\xi))d\xi \tag{13}$$

The modified algorithms is formulated as

$$u_{i+1}(x) = u_i(x) + \int_a^x \lambda(\xi)[Lu_i(\xi) - g(\xi) + \mathcal{L}^{-1}\sum_{i=0}^{i-1} u_i(\xi)]d\xi$$

$$u_{i+1}(x) = u_i(x) + \int_a^x (-1)^n \frac{1}{(i-1)!}(\xi - x)^{i-1}[Lu_i(\xi) - g(\xi) + \mathcal{L}^{-1}\int_a^{\xi} k(\xi, r)\sum_{i=0}^{i-1} u_i(\xi))dr]d\xi \tag{14}$$

Numerical Examples

In this section, modified variational iteration method is used to solve some non- linear Volterra integro-differential equations of the second kind.These numerical results will be compared with some other methods.

Example 1: Consider the nonlinear Volterra Integro-Differential equation (Nadjafi and Ghorbani, 2009)

$$u'(x) = 1 + \int_0^x u(t)u'(t)dt,\quad \text{for } x \in [0,1]. \tag{15}$$

The exact solution is $u_e(x) = \sqrt{2}\tan\left(\frac{x}{\sqrt{2}}\right)$. The correction functional for (15) is

$$u_{i+1}(x) = u_i(x) + \int_a^x \lambda(\xi)[Lu_i(\xi) - 1 - \int_a^{\xi}\mathcal{L}^{-1}\sum_{j=0}^{i}\tilde{u}_j(r)\tilde{u}_j'(r))dr]d\xi$$

Making the functional stationary and noting that $\tilde{u}_i$ is a restriction variation with $\delta\tilde{u}_i = 0$. So, (4) and (5) give

$$1 + \lambda|_{\xi=x} = 0 \quad \text{and} \quad \lambda'\big|_{\xi=x} = 0$$

The Lagrange's multiplier can be identified as $\lambda = -1$

$$u_{i+1}(x) = u_i(x) - \int_a^x [Lu_i(\xi) - 1 - \int_a^{\xi}\mathcal{L}^{-1}\sum_{j=0}^{i}\tilde{u}_j(r)\tilde{u}_j'(r))dr]d\xi. \tag{16}$$

From (6) $u'(x) - 1 = \int_0^x u'(t)u(t)dt$, setting $\int_0^x u'(t)u(t)dt = 0$

Let $L(u) = u'_0(x) - 1 = 0$

Using (14), the zeroth approximation is

$$u_0(x) = x. \tag{17}$$

Other approximations can be obtained as

$$u_1(x) = x + \frac{1}{6}x^4. \tag{18}$$

$$u_2(x) = x + \frac{1}{6}x^4 + \frac{1}{36}x^7 + \frac{1}{648}x^{10}$$

$$\begin{aligned}u_3(x) = {} & x + \frac{1}{6}x^4 + \frac{1}{30}x^5 + \frac{17}{2520}x^7 + \frac{31}{22680}x^9 + \frac{691}{2494800}x^{11} \\ & + \frac{1999}{35615657}x^{13} + \frac{79}{6945845}x^{15} + \frac{8}{3471025}x^{17} \\ & + \frac{1}{17635968}x^{22}\end{aligned}$$

Table 1: Computations showing comparison of some results with MVIM on Example 1

x_i	Exact Solution	Power Series Method (PSM)		Compact Finite Difference Method (CFDM) [15]		Homotopy Perturbation Method (HPM) [4, Nadjafi and Ghorbani, 2009]		Modified Variational Iteration Method (MVIM) n=3	
		Approximate Solution	Absolute Error	Approximate Solution	Absolute Error	Approximate Solution	Absolute Error	Approximate Solution	Absolute Error
0.0000	0.00000	0.00000	0.00E+00	0.00000	0.00E+00	0.00000	0.00E+00	0.00000	0.00E+00
0.0625	0.06254	0.06254	7.22E-07	0.06254	-4.00E-05	0.06254	1.00E-06	0.06254	0.00E+00
0.1250	0.12533	0.12532	6.54E-06	0.12533	-3.30E-04	0.12533	-3.00E-06	0.12533	0.00E+00
0.1875	0.18861	0.18860	6.41E-06	0.18860	-1.10E-03	0.18861	-4.00E-06	0.18861	0.00E+00
0.2500	0.25264	0.25263	7.14E-06	0.25263	-2.63E-03	0.25264	-3.00E-06	0.25264	0.00E+00
0.3125	0.31769	0.31777	8.04E-05	0.31769	-5.19E-03	0.31769	-2.00E-06	0.31769	0.00E+00
0.3750	0.38404	0.38404	3.50E-06	0.38403	-9.03E-03	0.38403	1.30E-05	0.38404	3.94E-15
0.4375	0.45201	0.45201	2.52E-06	0.45198	-1.45E-02	0.45199	2.30E-05	0.45201	7.33E-14
0.5000	0.52193	0.52193	5.15E-07	0.52187	-2.19E-02	0.52188	5.10E-05	0.52193	9.38E-13
0.5625	0.59417	0.59416	8.64E-06	0.59404	-3.15E-02	0.59405	1.19E-04	0.59417	8.92E-12
0.6250	0.66914	0.66914	1.93E-06	0.66890	-4.39E-02	0.6689	2.42E-04	0.66914	6.71E-11
0.6875	0.74732	0.74731	9.80E-06	0.74684	-5.93E-02	0.74685	4.70E-04	0.74732	4.18E-10
0.7500	0.82924	0.82923	8.97E-06	0.82837	-7.84E-02	0.82838	8.59E-04	0.82924	2.23E-09
0.8125	0.91552	0.91551	9.67E-06	0.91400	-1.02E-01	0.91401	1.51E-03	0.91552	1.04E-08
0.8750	1.00689	1.00687	1.62E-05	1.00432	-1.29E-01	1.00433	2.56E-03	1.00689	4.37E-08
0.9375	1.10419	1.10416	3.32E-05	1.10002	-1.63E-01	1.10002	4.17E-03	1.10419	1.67E-07
1.0000	1.20846	1.20838	8.02E-05	1.20184	-2.02E-01	1.20185	6.61E-03	1.20846	5.86E-07

Table 2: More computations showing comparison of results with MVIM on Example 1

x_i	Exact Solution	Adomian Decomposition Method (ADM) n=3		Conventional Variational Iteration Method (CVIM) n=3		Modified Variational Iteration Method (MVIM) n=3	
		Approximate Solution	Absolute Error	Approximate Solution	Absolute Error	Approximate Solution	Absolute Error
0.00000	0.00000	0.000000	0.00E+00	0.00000	0.00E+00	0.00000	0.00E+00
0.09380	0.09394	0.093806	1.31E-04	0.09381	1.31E-04	0.09394	0.00E+00
0.21880	0.22056	0.218991	1.57E-03	0.21899	1.57E-03	0.22056	0.00E+00
0.31250	0.31769	0.313298	4.39E-03	0.31330	4.39E-03	0.31769	0.00E+00
0.40620	0.41775	0.408491	9.26E-03	0.40849	9.26E-03	0.41775	1.79E-14
0.50000	0.52193	0.505303	1.66E-02	0.50530	1.66E-02	0.52193	9.38E-13
0.62500	0.66914	0.638177	3.10E-02	0.63818	3.10E-02	0.66914	6.71E-11
0.71880	0.78784	0.742299	4.55E-02	0.74230	4.55E-02	0.78784	9.84E-10
0.81250	0.91552	0.851853	6.37E-02	0.85185	6.37E-02	0.91552	1.04E-08
0.90620	1.05466	0.969144	8.55E-02	0.96914	8.55E-02	1.05466	8.62E-08
1.00000	1.20846	1.097370	1.11E-01	1.09737	1.11E-01	1.20846	5.86E-07

Example 2: Find the numerical solution for the first order Non-linear volterra integro differential equation

$u'(x) = -\frac{1}{2} + \int_0^x u'^2(\mathrm{t})\mathrm{dt}, \quad x \in [0,1],$ (19)

The exact solution is $u_e(x) = -\ln\left(\frac{x}{2}+1\right)$. The correction functional for (19) is

$\mathrm{u}_{i+1}(\mathrm{x}) = \mathrm{u}_i(\mathrm{x}) + \int_{\mathrm{a}}^{\mathrm{x}} \lambda(\xi)[Lu_i(\xi) + \frac{1}{2} - \int_{\mathrm{a}}^{\xi} \mathcal{L}^{-1} \sum_{j=0}^{i} (\tilde{u}_j')^2(\mathrm{r}))\mathrm{dr}]\mathrm{d}\xi$ (20)

Making the functional stationary and noting that, $\tilde{u}_n$ is a restriction variation with $\delta\tilde{u}_n = 0$. So, (4), and (5) give

$\delta\mathrm{u}_{i+1} = \delta\mathrm{u}_i + \delta\lambda(\xi)u_i(\xi) - \int_a^x \lambda'(\xi)\delta(u_i(\xi))\mathrm{d}\xi$

$\delta\mathrm{u}_{i+1} = \delta u_i(\xi)(1 + \lambda|_{\xi=x}) - \int_a^x \lambda'\delta u_i \mathrm{d}\xi$

To find the optimal $\lambda(\xi)$ and calculate variation with respect to u_i, we have the stationary conditions

$\lambda' = 0 \quad ; \quad 1 + \lambda|_{\xi=x} = 0$ (21)

Using (21) as a natural conditions, the Lagrange's multiplier is $\lambda = -1$

$\mathrm{u}_{i+1}(\mathrm{x}) = \mathrm{u}_i(\mathrm{x}) - \int_{\mathrm{a}}^{\mathrm{x}} [Lu_i(\xi) + \frac{1}{2} - \int_{\mathrm{a}}^{\xi} \mathcal{L}^{-1} \sum_{j=0}^{i} (\tilde{u}_j')^2(\mathrm{r}))\mathrm{dr}]\mathrm{d}\xi$

Let $L(u) = \mathrm{u}'_0(x) + \frac{1}{2} = 0$

Using (14), the zeroth approximation is

$\mathrm{u}_0(x) = -\frac{1}{2}x$

Other approximations are (22)

$$\mathrm{u}_1(x) = -\frac{1}{2}x + \frac{1}{8}x^3$$

$$\mathrm{u}_2(x) = -\frac{1}{2}x + \frac{1}{8}x^3 - \frac{1}{32}x^5 + \frac{3}{640}x^7$$

$$\mathrm{u}_3(x) = -\frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{24}x^3 + \frac{1}{64}x^4 - \frac{1}{240}x^5 + \frac{1}{1152}x^6 - \frac{1}{8064}x^7 - \frac{1}{129024}x^8 - \frac{1}{146080}x^9 + \frac{1}{460800}x^{11} - \frac{1}{901120}x^{13} + \frac{1}{1064960}x^{15}$$

Table 3: Computations showing comparison of conventional VIM with MVIM

$\mathbf{x}_i$	Exact	Conventional Variation Iteration Method (VIM)		MVIM	
		Approximate n=5	Error	Approximate n=3	Error
0.0	0.000000000	0.000000000	0.00E+00	0.000000	0.00E+00
0.1	-0.048790164	-0.048790144	-1.92E-08	-0.048790	-1.92E-08
0.2	-0.095310180	-0.095309613	-5.67E-07	-0.095310	-5.67E-07
0.3	-0.139761942	-0.139757965	-3.98E-06	-0.139758	-3.98E-06
0.4	-0.182321557	-0.183059759	-1.56E-05	-0.182306	-1.56E-05
0.5	-0.223143551	-0.223099354	-4.42E-05	-0.223099	-4.41E-05
0.6	-0.262364264	-0.262261841	-1.02E-04	-0.262262	-1.02E-04
0.7	-0.300104592	-0.299898036	-2.07E-04	-0.299899	-2.05E-04
0.8	-0.336472237	-0.336095817	-3.76E-04	-0.336099	-3.73E-04
0.9	-0.371563556	-0.370928469	-6.35E-04	-0.370937	-6.26E-04
1.0	-0.405465108	-4044565353	-8.16E-12	-0.404477	-9.88E-04

Example 3: Find the approximate solution of the first order Non-linear volterra integro differential equation of the convolution (Maleknejad *et al.*, 2011)

$u'(x) = -2\sin(\mathrm{x}) - \frac{1}{3}\sin(\mathrm{x}) - \frac{2}{3}\cos(2\mathrm{x}) + \int_0^x \cos(\mathrm{x}-\mathrm{t})u^2(\mathrm{t})\mathrm{dt}, \quad u(0) = 1$ (23)

The exact solution is $u_e(x) = \cos(\mathrm{x}) - \sin(x)$

The correction functional for (23) is

$$\mathrm{u}_{i+1}(\mathrm{x}) = \mathrm{u}_i(\mathrm{x}) + \int_{\mathrm{a}}^{\mathrm{x}} \lambda(\xi)[Lu_i(\xi) + 2\sin(\xi) + \frac{1}{3}\sin(\xi) + \frac{2}{3}\cos(2\xi) - \int_{\mathrm{a}}^{x} \mathcal{L}^{-1} \sum_{j=0}^{i} \cos(\xi - \mathrm{r})\tilde{\mathrm{u}}_j^2(\mathrm{r}))\mathrm{dr}]\mathrm{d}\xi$$

Making the functional stationary and noting that $\tilde{u}_n$ is a restriction variation with $\delta\tilde{u}_n = 0$. Then (4) and (5) give

$1 + \lambda|_{\xi=x} = 0$ and $\lambda'|_{\xi=x} = 0$

The Lagrange's multiplier is obtained as $\lambda = -1$

$$u_{i+1}(x) = u_i(x) - \int_a^x [Lu_i(\xi) + 2\sin(\xi) + \frac{1}{3}\sin(\xi) + \frac{2}{3}\cos(2\xi) - \int_a^x \mathcal{L}^{-1} \sum_{j=0}^{i} \cos(\xi - r)\tilde{u}_j^2(r))dr]d\xi \qquad (24)$$

Using the initial condition, the zeroth approximation is
$u_0(x) = 1$

Consequently, we have the following approximations

$$u_1(x) = 1 - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{1}{1008}x^7 + \frac{1}{20160}x^8 - \frac{1}{40320}x^9$$

$$u_2(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{5}{5040}x^7 + \frac{1}{20160}x^8 - \frac{1}{40320}x^9 - \frac{1}{192}x^{10} + \frac{1}{18900}x^{11} + \frac{1}{33600}x^{12} - \frac{1}{3628800}x^{13} - \frac{1}{907200}x^{14} + \frac{1}{2419200}x^{15} + \frac{13}{7257600}x^{16} - \frac{1}{812851200}x^{17} - \frac{1}{406425600}x^{18} + \frac{1}{1625702400}x^{19}$$

$$u_3(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \frac{5}{5040}x^7 + \frac{1}{20160}x^8 - \frac{1}{40320}x^9 - \frac{1}{60480}x^{10} + \frac{1}{151200}x^{11} + \frac{1}{302400}x^{12} - \frac{1}{3628800}x^{13} - \frac{4619447930799616191841 3}{1814400}x^{14} + \frac{1}{2419200}x^{15} + \frac{1}{108864000}x^{16}$$

Table 4: Computations showing comparison of some results with MVIM on Example 3

x_i	Exact	Compact Finite Difference Method [15]		Legendre wavelet method (LWM) (Maleknejad *et al.*, 2011; 13]		MVIM	
		Approximate	Error	Approximate	Error	Approximate	Error
0.0	1.000000000	0.999999	1.00E-06	1.000000	1.00E-06	1.000000	0.00E+00
0.1	0.895170749	0.895186	-1.53E-05	0.895171	-1.53E-05	0.895171	2.01E-11
0.2	0.781397247	0.781653	-2.56E-04	0.781397	-2.56E-04	0.781397	2.60E-09
0.3	0.659816282	0.659732	8.43E-05	0.659816	8.43E-05	0.659816	4.50E-08
0.4	0.531642652	0.530699	9.44E-04	0.531642	9.44E-04	0.531642	3.41E-07
0.5	0.398157023	0.398169	-1.20E-05	0.398155	-1.20E-05	0.398155	1.64E-06
0.6	0.260693142	0.260969	-2.76E-04	0.260687	-2.76E-04	0.260687	5.94E-06
0.7	0.120624500	0.120671	-4.65E-05	0.120607	-4.65E-05	0.120607	1.77E-05
0.8	-0.020649382	-0.020638	-1.14E-05	-0.020695	-1.14E-05	-0.020695	4.54E-05
0.9	-0.161716941	-0.161638	-7.89E-05	-0.161821	-7.89E-05	-0.161821	1.04E-04
1.0	-0.301168679	-0.301983	8.14E-04	-0.301389	8.14E-04	-0.301389	2.20E-04

Conclusion

This paper demonstrated the applicability of the Modified variational iteration method for approximating solution of non-linear Volterra- integro-differential equation of the second kind. The numerical results show that:

- The method provides a sequence of functions which converges to the exact solution of the problem by using three iterative steps without discretization of the variables.
- This method reduces the computational difficulty in solving non-linear volterra Integro-differential equations of the second kind when compared to other traditional methods.
- The method is promising and readily implemented which makes it more efficient tool and more practical for solving non-linear integro-differential equations.

Conflict of Interest

The authors declare that there is no conflict of interest related to this study.

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